# A Treatment of Quantum Electrodynamics as a Model of Interactions between Sizeless Particles in Relational Quantum Gravity

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**Abstract**:
*A Relational Quantum Theory Incorporating Gravity* developed the concept of quantum covariance and argued that this is the correct expression of the fundamental physical principle that the behaviour of matter is everywhere the same in the quantum domain, as well as being the required condition for the unification of general relativity with quantum mechanics for non-interacting particles. This paper considers the interactions of elementary particles. Quantum covariance describes families of finite dimensional Hilbert spaces with an inbuilt cut-off in energy-momentum and using flat space metric (*quantum coordinates*) between initial and final states. It is shown by direct construction that it is possible to construct a quantum field theory of operators on members of these families, obeying locality, suitable for a description of particle interactions, and leading to a general formulation of particle theoretic field theory incorporating qed. The construction is consistent and effectively identical in the continuum limit to classical quantum electrodynamics with all loop divergences removed by the method of Epstein and Glaser. The model avoids the Landau pole. The model does not start from a classical Lagrangian and it is shown that the interaction leads to Maxwell's equations in the classical limit and to Feynman rules as in the standard theory after renormalisation.

## 1 Introduction

### 1.1 Background

*A Relational Quantum Theory Incorporating Gravity* (RQG) [6] is a formal development of a relational quantum theory from fundamental principles of measurement. In that treatment states in Hilbert space are not regarded as physical states of matter, but as statements in quantum logic, a formal language about measurement results, including statements in the subjunctive mood. Reality is assumed to exist independent of observation, but measurement is a relationship between matter and matter, and the quantum theory describes what may be known to an observer, not the absolute state of matter. The model has a number of subtle differences from standard quantum theory, and is background free in the sense that the physical metric is a relationship resulting from interactions between particles, not a prior property of space or spacetime. Standard general relativity is found in the classical correspondence, except that in the quantum domain the affine connection is replaced with a teleconnection, defined remotely between coordinate systems used to describe the initial and final states in the quantum theory. This has implications for cosmological redshift when the emission of a photon from a distant object and detection on earth are regarded as initial and final states in the quantum theory. The teleconnection yields a number of verified predictions in Cosmology and provided accounts of phenomena for which the standard model has no explanation [7].

Particle theoretic qed has been largely out of favour for more than half a century (Feynman being a notable exception [18]), but there have been many developments in our understanding during that time. Among the problems particle qed has to face are the requirement of a positive definite norm for valid probabilities, the indefinability of the equal point multiplication between field operators, loop divergences, and the Landau Pole. The purpose of the present paper is to review the particle theoretic, or Fock space, formulation of quantum electrodynamics and to show by explicit construction that a rigorous formulation exists as a non-perturbative model, based in relational quantum theory and having no divergent quantities. The treatment does not start with a classical Lagrangian or with the quantisation of a classical field. Instead, a Hilbert space of particle states is used, together with an interaction between sizeless particles. Many physicists are satisfied that qed gives correct predictions and this construction of qed does not alter these. The requirement for a formal construction from first principles is essentially mathematical and philosophical, but while the mathematical construction is of primary concern attention will also be paid to interpretation and physical considerations will be used to shape and motivate the mathematical model.

### 1.2 Quantum Covariance

Clearly field operators acting on finite dimensional Hilbert space cannot be manifestly covariant. The use of a finite dimensional Hilbert space in RQG reflects the fact that a physical measuring apparatus always has a finite range and resolution. Basis states in Hilbert space are defined with respect to the apparatus used to determine them. Then rotating the reference frame means rotating the apparatus and manifest covariance does not apply. Quantum covariance considers the set $\mathcal{N}$ of finite, discrete coordinate systems, $\mathrm{N_T} \subset \mathbb{R}^4$, which can be defined in principle by a measurement apparatus. Each coordinate system defines a basis for a Hilbert space at any given time. In the quantum covariant formulation the wave function is defined on a continuum, while the inner product is discrete; in a change of reference frame, the lattice and inner product used in one Hilbert space replaced with the lattice and inner product of another.

It will be clear from the use of a family of finite dimensional Hilbert spaces that the present construction does not obey the Wightman axioms, which specify a single covariant Hilbert space. However the model obeys axioms as strong as those cited by Wightman [19] and by Osterwalder and Schrader [14]. Quantum covariance does not require the limit of small lattice spacing or large lattice size. However, loop integrals are shown to be cut-off independent to first order so the continuum limit exists. Manifest covariance and the "renormalised" formulae of the standard model are recovered in this limit, although no infinite renormalisation is required. The Landau pole does appear in the limit as the discrete time interval tends to zero, but not if lattice spacing is bounded below by a fundamental interval of proper time in the interactions between particles. This appears to be the case, since in RQG a minimum proper time between interactions was shown to have a geometrical import equivalent to Einstein's field equation. In the continuum limit the algebras of operators defined by (4.1.18) for the photon, and by (4.2.1) together with (4.2.5) and (4.2.8) for the Dirac field, obey the Haag-Kastler axioms given in [8]. If quantum covariance is required, instead of general or Lorentz covariance, the Haag-Kastler axioms are satisfied without taking the limit.

### 1.3 Differences from Other Models

Instead of starting with a mathematical theory and trying to interpret it, RQG adopts an orthodox interpretation and seeks to produce the mathematical structure appropriate to it. This leads to a particulate quantum theory with subtle differences from standard quantum mechanics. Despite the appearance of a lattice, there are important differences between RQG and lattice quantum field theory as developed by Wilson and others (e.g. Montvay & Münster [13]). For example, RQG uses Minkowski rather than Euclidean co-ordinates, and has a bounded momentum space with an automatic cut-off. An energy cut-off follows from the mass shell condition, but there is no cut-off in the off mass shell energy which appears in the perturbation expansion, since this is an abstract parameter in a contour integral ((5.3.2) to (5.3.5)), not a physical value. Using finite dimensional Hilbert space, fields are operator valued functions, not distributions. The field operators constructed here will be used to describe interactions between particles using the interaction picture. Although formally similar to quantised free fields, they describe the potential for the creation or annihilation of a particle in an interaction. This cannot be reconciled to interpretational statements which are sometimes made about models which arise from the quantisation of classical fields, such as "*The free field describes particles which do not interact*" (Glimm & Jaffe p 100 [8]) or *"In its mature form, the idea of quantum field theory is that quantum fields are the basic ingredients of the universe, and particles are just bundles of energy and momentum of the fields"* (Weinberg [21] ). While the formulae of the present construction are extremely like the formulae which arise in such models and give the same physical predictions in the appropriate limits, the interpretation of these formulae is quite different.

A significant mathematical difference between this and other formulations of relativistic field theory is that time is a parameter, as in non-relativistic quantum mechanics, and each Hilbert space $\mathbb{H}(t)$ is formulated for synchronous states at time $t$. The $U$-matrix is a map $U(t_1, t_2):\mathbb{H}(t_1) \to \mathbb{H}(t_2)$ where $t_1 \neq t_2$, so that unitarity does not apply (conservation of probability is imposed separately). Homomorphically identifying $\mathbb{H}(t) = \mathbb{H}$ for all $t$ introduces what has been one of the central problems in the construction of quantum field theory in 4 dimensions, namely the indefinability of the equal point multiplication between the field operators (which is definable in a finite dimensional space but not in the limit as lattice spacing goes to zero). This is resolved pragmatically by normal ordering all equal point products of field operators. This non-linear condition is motivated physically by saying that a particle created at $x_0^n$ cannot interact again at the same instant. Thus we will assert that the correct description of physical processes in qft uses an interaction Hamiltonian $H(x) \neq 0$ such that $H^2(x) = 0$ (nilpotency does not apply because equal time products of field operators are normal ordered; otherwise $H^2(x)$ would diverge in the continuum limit). In other respects $H$ is much as in ordinary qed. Linearity of time evolution is required to avoid a dependency on histories; the interactions of a particle created yesterday should be no different from those of one created the day before. Normal ordering equal time products of fields is a non-linearity, but the time evolution proposed here is linear for states created in the past, and only distinguishes the interaction of a particle created "now" from those of particles created "not now", by imposing the condition that a particle cannot interact twice in one instant, or equivalently, that $H(x)$ cannot act on the result of itself.

This argument gives physical motivation to the method of Epstein & Glaser [17][3] in which $\theta(t_1 - t) - \theta(t - t_1)$ is replaced with a continuous switching function which is zero at $t = t_1$, removing loop divergences. In the continuum limit the analysis of the origin of the ultraviolet divergence is, for practical purposes, that given by Scharf [17], namely the incorrect use of Wick's theorem. The difference between this treatment and Scharf is that here the limiting procedure uses a discrete sum whereas Epstein and Glaser use a continuous switching function, and while Scharf says (p163) “the switching on and off the interaction is unphysical”, here the switching off and on of the interaction at $x_0^n = x_0^j$ is regarded as a physical constraint meaning that only one interaction takes place for each particle in any instant. As with the method of Epstein & Glaser, this leads directly to a perturbation expansion in which the terms are finite and similar in form to the standard “renormalised” expansion.

### 1.4 Outline

Based on the quantum covariant formulation of quantum mechanics the construction proceeds under broadly conventional lines. Section 2, *Particles*, introduces spin and reviews the photon and the Dirac particle. Section 3, *Field Theory*, discusses interactions and, after defining creation and annihilation operators and field operators, establishes conservation of 3-momentum in interactions. Section 4, *Particle Fields*, defines the photon field and the Dirac field. There is no assumption of a Lagrangian or of classical laws, but section 5, *Electrodynamics*, establishes that Maxwell’s equations follow from the simple interaction in which a Dirac particle emits or absorbs a photon, showing that the renormalised mass and coupling constant are equal to their bare values in the low energy limit. Feynman rules are calculated and there is no first order cut-off dependency in the perturbation expansion, the predictions being those of the standard renormalised theory by the method of Epstein and Glaser [3][17].

This paper is a continuation of RQG and uses many of the same notations, but some minor notational changes will be adopted. The space-time coordinate system can be taken as, $\mathrm{N_T} \equiv (-\nu, \nu\,]^4 \subset \mathbb{N}^4$. Bold type will be used for 3-vectors. In RQG coordinate space vectors are denoted with a bar. Here, for the sake of notational convenience, the bars have been omitted. In flat space, coordinate space vectors are equal to physical vectors, and the present treatment applies without modification. When gravity is taken into account, the coordinate space vectors used in the quantum theory must be converted back into physical vectors in measurement.

The method for converting coordinate space vectors to physical vectors is straightforward. Unprimed conformally flat coordinates are defined with an origin at the initial measurement, in accordance with two way photon exchange (e.g. the radar method). Similarly primed conformally flat coordinates are defined with an origin at the final measurement. Coordinate space momentum $p$ in unprimed coordinates is defined equal to physical momentum in the initial state. For each event a coordinate space displacement vector $x$ is defined with components equal to the coordinates of the event in unprimed coordinates. The coordinate space vectors, $x$ and $p$, are then used to describe flat space wave motions in the unprimed coordinates. In the final measurement $x$ and $p$ are converted to primed coordinates. If cosmological redshift is neglected we have $x^{\alpha'} = x^{\alpha'}{}_{,\beta} x^{\beta}$ and $p^{\alpha'} = x^{\alpha'}{}_{,\beta} p^{\beta}$. Cosmological expansion introduces the redshift factor factor $1 + z = a_0^2 / a^2$, as described in RTG.

QED is studied here, but the treatment admits immediate extension to weak interactions and to QCD. RQG used eigenstates of position as a basis for Hilbert space. The treatment here extends that account. For fermions spin is required as well as position. Elementary Bosons are always created or destroyed in interaction and do not exist in eigenstates of position. The measured position of the interaction (when it exists) is the position of a fermion, not the position of a boson. As a result, for bosons the states $|x\rangle$ are introduced as an arbitrary basis and not directly related to measured states.

## 2 Particles

This section reviews Dirac particles and photons. Since wave functions are reference frame dependent there is no reason to normalise the momentum space wave function using the covariant integral. In other respects the treatment is standard and proofs will be omitted.

### 2.1 Equation of Motion

Conservation of probability requires unitary time evolution, which is equivalent to a first order differential equation in the form

$$i\partial_0 f = Hf \tag{2.1.1}$$

for some Hermitian operator $H$. The eigenstates of $i\partial_0$ are states of constant energy so that

$$i\partial_0 f = Ef \tag{2.1.2}$$

This is not covariant. A covariant equation would be an equation for proper time evolution, i.e. evolution in the rest frame of the particle with $p = (m, 0, 0, 0)$ This requires the scalar product between $P^\mu(0) = -i\partial^m$ and the wave function. It has the form, for some vector operator $\Gamma$ and real constant $\mu$

$$P_\mu(x)\Gamma^\mu f = -ik^\gamma{}_\mu(x)\partial_\gamma \Gamma^\mu f = \mu f \tag{2.1.3}$$

### 2.2 Spin

Following Dirac [2], there is no invariant equation in the form of (2.1.3) for scalar $f$ and the theory breaks down[1]. To rectify the problem a spin index is added to N, such that $\mathrm{N}_S = \mathrm{N} \otimes S$ where $S$ is a finite set of indices. When there is no ambiguity spin is suppressed, so that $\mathrm{N} = \mathrm{N}_S$, and the treatment described in RQG goes through as before. When we wish to make the spin index explicit we write $|x\rangle = |x, \alpha\rangle = |x\rangle_\alpha$ normalised to

$$\forall(x, \alpha), (y, \beta) \in \mathrm{N}_S \quad \langle x, \alpha|y, \beta\rangle = \langle x|y\rangle_{\alpha\beta} = \delta_{\boldsymbol{xy}}\delta_{\alpha\beta} \tag{2.2.1}$$

The wave function now has a spin index

$$f(x) = f_\alpha(x) = \langle x|f\rangle_\alpha \tag{2.2.2}$$

and the braket becomes

$$\langle g|f\rangle = \sum_{x \in \mathrm{N}_S} \langle g|x\rangle\langle x|f\rangle = \sum_{(x, \alpha) \in \mathrm{N}_S} \overline{g_\alpha(x)} f_\alpha(x) \tag{2.2.3}$$

It is now possible to find a covariant equation in the form (2.1.3), namely the Dirac equation,

$$i\partial \cdot \gamma f(x) = mf(x) \tag{2.2.4}$$

Another possibility is that $\mu = 0$, $f$ is a vector and $\Gamma = g$ the metric tensor. Then (2.1.3) reduces to

$$i\partial \cdot f(x) = 0 \tag{2.2.5}$$

Vector particles obeying (2.2.5) may have non-zero mass, and the treatment below admits straightforward adaptation to weak interactions, but without a prediction of Higgs. Only QED is studied here. (2.2.5) can be understood as the equation of motion of a vector particle which is only ever created or destroyed in interaction. This means that the discussion of quantum logic given in RQG does not apply directly to vector bosons. The observables associated with bosons are determined from measurements of position of Dirac particles, in the knowledge of the interaction between bosons and Dirac particles. The probability for detecting a photon at $x$ is not $|\langle x|f\rangle|^2$ but also depends on the photon field operator and on the Dirac $\gamma$ matrices. The norm is intended to generate physically realisable predictions of probability, and must be both invariant and positive definite. It is given by

$$\langle f|f\rangle = \sum_{(x, \alpha) \in \mathrm{N}_S} \overline{f_\alpha(x)} f_\alpha(x) \tag{2.2.6}$$

1. This applies to fundamental particles but does not preclude scalar composite or scalar ghost particles.

If $f$ transforms as a space-time vector, (2.2.6) is only invariant if (2.2.1) is replaced by the definition

$$\forall(x,\alpha),(y,\beta)\in N \quad \langle x,\alpha|y,\beta\rangle = \langle x|y\rangle_{\alpha\beta} = \eta(\alpha)\delta_{xy}\delta_{\alpha\beta} \tag{2.2.7}$$

where $\eta(0) = -1$ and $\eta(1) = \eta(2) = \eta(3) = 1$. The factor -1 will be implicit in summing the zeroeth index for vector spin, so that (2.2.3) and (2.2.6) are retained with $\eta$ suppressed. (2.2.7) is invariant, but not positive definite as required by a norm. The probability interpretation requires that any vector particles have a positive definite norm for observed states. So only space-like polarisation of vector particles can be observed. Other polarisation states are permitted and are required for the derivation of Maxwell's equations in section 5, *Electrodynamics*, but we can only observe states with positive norm.

### 2.3 Dirac Particles

Bold type will be adopted for 3-vectors. The solution to the Dirac equation (2.2.4) is

$$f_\alpha(x) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\sum_{r=1}^{2}\int_{\mathbb{R}^3} d^3\boldsymbol{p}\; F(\boldsymbol{p},r)u_\alpha(\boldsymbol{p},r)\; e^{-ix\cdot p} \tag{2.3.1}$$

where $p$ satisfies the mass shell condition and $u$ is a Dirac spinor, having the form

$$u(\boldsymbol{p},r) = \sqrt{\frac{p_0+m}{2p_0}}\begin{bmatrix}\zeta(r)\\ \dfrac{\sigma\cdot\boldsymbol{p}}{p_0+m}\zeta(r)\end{bmatrix} \qquad \text{for } r = 1,2 \tag{2.3.2}$$

where $\zeta$ is a two-spinor normalised so that

$$\bar{\zeta}_\alpha(r)\zeta_\alpha(s) = \delta_{rs} \tag{2.3.3}$$

where the summation convention is used for repeated spin indices, and $\sigma = (\sigma_1,\sigma_2,\sigma_3)$ are the Pauli spin matrices. In this normalisation $\bar{u}_\alpha(\boldsymbol{p},r)u_\alpha(\boldsymbol{p},s) = \delta_{rs}$. $F(\boldsymbol{p},r)$ is the momentum space wave function given by inverting (2.3.1) at $x_0 = 0$

$$F(\boldsymbol{p},r) = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}}\sum_{(\boldsymbol{x},\alpha)\in N} f_\alpha(0,\boldsymbol{x})\bar{u}_\alpha(\boldsymbol{p},r)e^{i\boldsymbol{x}\cdot\boldsymbol{p}}\,. \tag{2.3.4}$$

**Definition:** *Using Dirac $\gamma$-matrices as defined in the literature the Dirac adjoint is* $\hat{u} = \bar{u}\gamma^0$

### 2.4 Antiparticles

The treatment of the antiparticle modifies the Stückelberg-Feynman [20],[4] interpretation by considering the mass shell condition. The Dirac equation is most readily understood as the equation of motion for a particle in its own proper time. If every particle has its own proper time, and if there is no other fundamental time, then it is natural to think that one particle's proper time can be reversed compared to another; antimatter is matter whose proper time is inverted compared to surrounding matter. A sign is lost in the mass shell condition, due to the squared terms, but a time-like vector with a negative time-like component provides a natural definition of $m < 0$. So permissible solutions of the Dirac equation, (2.2.4), have positive energy $E = p^0 > 0$ when $m$ is positive and negative energy when $m$ is negative. Complex conjugation reverses time while maintaining the probability interpretation and restores positive energy, and we also change the sign of mass, $m \to -m$. Thus the negative energy solution is transformed and satisfies

$$i\partial\cdot\bar{\gamma}f(x) = -mf(x) \tag{2.4.1}$$

where $\gamma$ is the complex conjugate, $\bar{\gamma}^j_{\alpha\beta} = \overline{\gamma^j_{\alpha\beta}}$. Although this is slightly different from the positron wave function cited in e.g.[1] the treatments will be reconciled in the definition of the field operators.

The solution to (2.4.1) is the wave function for the antiparticle

$$f(x) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \sum_{r=1}^{2} \int_{\mathbb{R}^3} d^3\boldsymbol{p}\ F(\boldsymbol{p}, r)\bar{v}(\boldsymbol{p}, r)\ e^{-ix \cdot p} \tag{2.4.2}$$

where $p$ satisfies the mass shell condition, and $\bar{v}$ is the complex conjugate of the Dirac spinor.

$$v(\boldsymbol{p}, r) = \sqrt{\frac{p_0 + m}{2p_0}} \begin{bmatrix} \frac{\sigma \cdot \boldsymbol{p}}{p_0 + m}\zeta(r) \\ \zeta(r) \end{bmatrix} \qquad \text{for } r = 1,2$$

The spinor has the normalisation $\bar{v}_\alpha(\boldsymbol{p}, r)v_\alpha(\boldsymbol{p}, s) = \delta_{rs}$. $F(\boldsymbol{p},r)$ is the momentum space wave function given by

$$F(\boldsymbol{p}, r) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \sum_{(\boldsymbol{x}, \alpha) \in \mathrm{N}} f_\alpha(0, \boldsymbol{x})v_\alpha(\boldsymbol{p}, r)e^{i\boldsymbol{x} \cdot \boldsymbol{p}} \tag{2.4.3}$$

## 2.5 The Photon

The wave function for the photon is the solution to the equation of motion, (2.2.5), together with the Klein-Gordon equation with zero mass

$$f_\alpha(x) = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \int_{\mathbf{R}^3} d^3\boldsymbol{p}\ F(\boldsymbol{p}, r)w_\alpha(\boldsymbol{p}, r)\ e^{-ix \cdot p} \tag{2.5.1}$$

where

i. $p^2 = 0$

ii. $w$ are orthonormal vectors given by

a) $w(\boldsymbol{p}, r) = (1, \boldsymbol{0})$

b) for $r = 1,2,3$, $w(\boldsymbol{p}, r) = (0, \boldsymbol{w}(\boldsymbol{p}, r))$ are such that $\boldsymbol{w}(\boldsymbol{p}, 3) = \boldsymbol{p}/p_0$ is longitudinal and $\boldsymbol{w}(\boldsymbol{p},1)$ and $\boldsymbol{w}(\boldsymbol{p},2)$ are transverse with $\boldsymbol{w}(\boldsymbol{p}, r) \cdot \boldsymbol{w}(\boldsymbol{p}, s) = \delta_{rs}$

iii. $F$ is such that observable states of the photon cannot be polarised in the longitudinal and time-like spin states, i.e. in any state such that the annihilation of a photon can be detected,

$$F(\boldsymbol{p},0) = F(\boldsymbol{p},3) \tag{2.5.2}$$

$F(\boldsymbol{p},r)$ is the momentum space wave function given by inverting (2.5.1) at $x_0 = 0$

$$F(\boldsymbol{p}, r) = \left(\frac{\chi}{2\pi}\right)^{\frac{3}{2}} \eta(r) \sum_{(\boldsymbol{x}, \alpha) \in \mathrm{N}} f_\alpha(0, \boldsymbol{x})w_\alpha(\boldsymbol{p}, r)e^{i\boldsymbol{x} \cdot \boldsymbol{p}} \tag{2.5.3}$$

## 2.6 Plane Wave States

**Definition:** *Plane wave states* $|\boldsymbol{p}, r\rangle = \mathbb{H}$ *are defined by the wave functions* $\forall x \in \mathbb{R}^4$

$$f(x) = \langle x | \boldsymbol{p}, r \rangle = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} u(\boldsymbol{p}, r)e^{-ix \cdot p} \qquad \textit{for the Dirac particle} \tag{2.6.1}$$

$$f(x) = \langle x | \boldsymbol{p}, r \rangle = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \bar{v}(\boldsymbol{p}, r)e^{-ix \cdot p} \qquad \textit{for the antiparticle, and} \tag{2.6.2}$$

$$f(x) = \langle x | \boldsymbol{p}, r \rangle = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} w(\boldsymbol{p}, r)e^{-ix \cdot p} \qquad \textit{for the photon.} \tag{2.6.3}$$

**Theorem:** (Newton's first law) In an inertial reference frame an elementary particle in isolation has a constant momentum space wave function. $\forall | f \rangle \in \mathbb{H}$

$$| f \rangle = \sum_r \eta(r) \int_{\mathbb{R}^3} d^3\boldsymbol{p} |\boldsymbol{p}, r\rangle\langle \boldsymbol{p}, r | f \rangle \tag{2.6.4}$$

**Corollary:** The time evolution of the position function of a particle in isolation is, $\forall | f \rangle \in \mathrm{H}$,

$$\langle x|f\rangle = \sum_r \eta(r) \int_{\mathbb{R}^3} d^3\boldsymbol{p} \langle x|\boldsymbol{p}, r\rangle\langle \boldsymbol{p}, r|f\rangle \tag{2.6.5}$$

where $r$ = 0-3 for photons, and $r$ = 1-2 for Dirac particles ($\eta$ is redundant for a Dirac particle).

**Corollary:** The resolution of unity

$$\sum_r \eta(r) \int_{\mathbb{R}^3} d^3\boldsymbol{p} |\boldsymbol{p}, r\rangle\langle \boldsymbol{p}, r| = 1 \tag{2.6.6}$$

**Corollary:** The braket has the time invariant form

$$\langle g|f\rangle = \sum_r \eta(r) \int_{\mathbb{R}^3} d^3\boldsymbol{p} \langle g|\boldsymbol{p}, r\rangle\langle \boldsymbol{p}, r|f\rangle \tag{2.6.7}$$

**Theorem:** $\langle \boldsymbol{q}, s|\boldsymbol{p}, r\rangle$ is a delta function on the test space of momentum space wave functions

$$\langle \boldsymbol{q}, s|\boldsymbol{p}, r\rangle = \eta(r)\delta_{rs}\delta(\boldsymbol{p} - \boldsymbol{q}) \tag{2.6.8}$$

**Corollary:** The braket for the photon is positive definite as required by the probability interpretation, and (2.6.7) reduces to

$$\langle g|f\rangle = \sum_{r=1}^{2} \int_{\mathbb{R}^3} d^3\boldsymbol{p} \langle g|\boldsymbol{p}, r\rangle\langle \boldsymbol{p}, r|f\rangle \tag{2.6.9}$$

## 2.7 Gauge Invariances

It follows from (2.6.9) that the braket is invariant under the addition of a (non-physical) light-like polarisation state, so that light-like polarisation cannot be determined from experimental results. Although their value is hidden by gauge, the time-like and longitudinal polarisation states cannot be excluded because, as we will see, they contribute to the electromagnetic force. A second form of gauge invariance is found by letting $g$ be an arbitrary solution of $\partial^2 g = 0$. It is routine to show that observable results are invariant under gauge transformation of the photon wave function given by

$$f_\alpha(x) \to f_\alpha(x) + \partial_\alpha g(x) \tag{2.7.1}$$

The gauge term, $\partial_\alpha g$, has no physical meaning.

# 3 Field Theory

## 3.1 Interactions

In this treatment $\mathbb{H}$ is simply a naming system. Its construction required no physics beyond the knowledge that we can count particles, we can measure the position of individual particles, and that we can measure the relative frequency of each result of a repeated measurement. The description of physical processes in terms of $\mathbb{H}$ requires a law describing the time evolution of kets when interactions are taken into account. An interaction at time $t$ is described by an operator, $I(t)$. Hilbert space is defined at some time $t$, $\mathbb{H} = \mathbb{H}(t)$, so that the interaction is a map

$$I(t): \mathbb{H}(t) \to \mathbb{H}(t+1)$$

where $t$ is measured in units of the minimal proper time between interactions, which was introduced in RQG. For definiteness we may take

$$\forall \boldsymbol{x}^i \in \mathrm{N}, \forall n \in \mathrm{N}, \langle \boldsymbol{x}^1;\ldots;\boldsymbol{x}^n|I|\boldsymbol{x}^1;\ldots;\boldsymbol{x}^n\rangle = 0 \tag{3.1.1}$$

since otherwise there would be a component of $I$ corresponding to the absence of interaction. At each time, $t$, either no interaction takes place and the state $|f\rangle \in \mathbb{H}$ is unchanged or an interaction, $I$, takes place. By the identification of the operations of vector space with weighted

OR between uncertain possibilities, the possibility of an interaction at time $t$ is described by the map

$$|f\rangle \to \mu(1 - iI(t))|f\rangle$$

where $\mu$ is a scalar chosen to preserve the norm, as required by the probability interpretation. $|\mu| = 1$ by (3.1.1). Thus the law of evolution of the ket from time $t$ to time $t+1$ is

$$|f\rangle_{t+1} = \mu(1 - iI(t))|f\rangle_t \tag{3.1.2}$$

To first order (3.1.2) is identical to the time evolution equation $|f\rangle_{t+1} = \mu e^{-iI(t)}|f\rangle_t$ found in, for example, lattice field theory [13], but here (3.1.2) is interpreted literally as meaning that in each instant particle either interacts or does not interact.

Since $I(t)$ is a map from one Hilbert space, $\mathbb{H}(t)$, defined at time $t$ to another, $\mathbb{H}(t+1)$, defined at $t+1$, we cannot talk of it being self adjoint or of the spectrum of $1 - iI(t)$. If we identify the Hilbert spaces at different times using the natural homomorphism defined by the basis of position kets, then we cannot use a linear operator, $I(t)$, on the resultant space to physically represent an interaction. Linearity is normally imposed since when $I(t)$ acts on a state prior history should not be relevant. But a careful analysis suggests that $I(t)$ is not strictly linear, because linearity would dictate that an the action of $I(t)$ on a particle created by $I(t)$ at time $t$ should be the same as its action on a particle previously created and evolving to the same ket, which would mean that a particle can physically interact twice in the same instant. This appears physically meaningless and $I(t)$ is regularised by imposing the condition that a particle cannot be annihilated at the instant of its creation. In other words $I(t)$ cannot act on the results of itself, and we have

$$I^2(t) = 0 \tag{3.1.3}$$

In other respects $I(t)$ is linear. There is no nilpotent solution to (3.1.3) for a linear operator $I(t)$ because the norm must be preserved to maintain the probability interpretation. This would lead to two constraints, that $I(t)$ is self-adjoint and $1 - iI(t)$ is unitary. But if $1 - iI(t)$ is unitary its spectrum lies on the unit circle, and if $I(t)$ is self-adjoint its spectrum lies on the real line. So, by the spectral mapping theorem, the spectrum of $1 - iI(t)$ lies on the line $\{1 - ix | x \in \mathrm{R}\}$. But the intersection of the unit circle with this line is just $\{1\}$ so $1 - iI(t)$ would be the identity operator, and we would have $I(t) = 0$. Physicists usually conclude is that there is no discrete model., but here the resolution is a non-linear condition to prohibit the physical annihilation of a particle at the instant of its creation

The removal of products describing the annihilation of a particle at the instant of its creation, as in 7.5 is most naturally done by normal ordering, but, whereas it is usual to normally order the interaction Hamiltonian, according to the argument above all equal time products of field operators are normal ordered. In practice this is largely academic, since it will be shown in section 5.3, *Feynman Rules* that terms containing $I^2(t)$ do not appear in the perturbation expansion. It will also be seen that the exclusion of these terms removes the cut-off dependency to first order and leads directly to a finite “renormalised” perturbation expansion in the continuum limit.

By (3.1.2) preservation of the norm implies that $\forall |f\rangle \in \mathrm{F}$

$$\langle f|f\rangle = \langle f|(1 + iI^\dagger)\bar{\mu}\mu(1 - iI)|f\rangle \tag{3.1.4}$$

$$= |\mu|^2(\langle f|f\rangle + \langle f|I^\dagger I|f\rangle + i\langle f|I^\dagger - I|f\rangle) \tag{3.1.5}$$

Normal ordering implies that $\langle f|I^\dagger I|f\rangle = 0$ So

$$\frac{\langle f|I^\dagger - I|f\rangle}{\langle f|f\rangle} = i\frac{1 - |\mu|^2}{|\mu|^2} \tag{3.1.6}$$

(3.1.6) has a straightforward solution with $|\mu|^2 = 1$ . and $I = I^\dagger$ . Although strictly non-linearity in normal ordering implies that $I$ is not Hermitian, $I^2(t)$ does not appear in the physical model, and $I$ may be treated as Hermitian and will be referred to as such.

### 3.2 Creation Operators

The creation of a particle in an interaction is described by the action of a creation operator. Creation operators incorporate the physical idea that when a particle is created it is impossible to distinguish it from any existing particle of the same type. There is some advantage in using creation and annihilation operators to (anti)symmetrise states, since this ties in with the idea that physical states are created in interactions which are themselves described as combinations of creation and annihilation operators. Creation operators are defined by their action on a basis. Ket notation will be used for creation operators as it will simplify the notations of field theory. This is justified by the homomorphism defined by (3.2.1). It will always be possible to distinguish states from creation operators by context

**Definition:** $\forall x \in \mathrm{N}$ *the creation operator* $|x\rangle:\mathbb{H}\to\mathbb{H}$ *is* $\forall y,\ y^i \in \mathrm{N}, i = 1, \dots, n$

$$|x\rangle|\,\rangle = |x\rangle \tag{3.2.1}$$

$$|x\rangle:|y\rangle \to |x\rangle|y\rangle = |x;y\rangle = \frac{1}{\sqrt{2}}[(|x\rangle,|y\rangle)+\kappa(|y\rangle,|x\rangle)] \tag{3.2.2}$$

$$|x\rangle:(|y^1\rangle,\dots,|y^n\rangle) \to \frac{1}{\sqrt{n+1}}\left((|x\rangle,|y^1\rangle,\dots,|y^n\rangle) + \kappa\sum_{i=1}^{n}(|y^i\rangle,|y^1\rangle,\dots,|x\rangle,\dots,|y^n\rangle)\right) \tag{3.2.3}$$

where $|x\rangle$ appears in the $i$+1th position in the $i$th term of the sum. It is routine to show that $\kappa = \pm 1$ for Bosons and Fermions respectively

**Definition:** *More generally creation operators are defined by linearity*

$$\forall\ |f\rangle \in \mathrm{H}^1 \qquad |f\rangle = \sum_{x\in\mathrm{N}}\langle x|f\rangle|x\rangle \tag{3.2.4}$$

**Definition:** *The space of (anti)symmetric states* $\mathbb{F}\subset\mathbb{H}$ *is generated from* $\mathbb{H}^0 = \{|\,\rangle\}$ *by creation operators. Physical states are elements of* $\mathbb{F}$.

**Definition:** *Notation for elements of* $\mathbb{F}$ *is defined inductively:*

$$\forall\ |g\rangle \in \mathbb{H}^1, \forall\ |f\rangle \in \mathbb{H}^n \cap \mathbb{F}\ ,\ |g;f\rangle = |g\rangle|f\rangle \in \mathbb{H}^{n+1}\cap\mathbb{F} \tag{3.2.5}$$

### 3.3 Annihilation Operators

In an interaction particles may be created, as described by creation operators, and particles may change state or be destroyed. The destruction of a particle in interaction is described by the action of an annihilation operator. A change of state of a particle can be described as the annihilation of one state and the creation of another. Thus a complete description of any process in interaction can be achieved through combinations of creation and annihilation operators.

**Definition:** $\forall|f\rangle\in\mathbb{H}^1$*, the annihilation operator* $\langle f|:\mathbb{F}\to\mathbb{F}$ *is the Hermitian conjugate of the creation operator* $|f\rangle:\mathbb{F}\to\mathbb{F}$, $\langle f| = |f\rangle^\dagger$ .

### 3.4 Field Operators

**Definition:** *A partial field* $\Psi = \Psi(\mathrm{N})$ *is a family of mappings* $\Psi(\mathrm{N}): \mathbb{R}^4 \otimes S \to \mathbb{F}$*, where S is the set of spin indices and the elements of* $\mathbb{F}$ *are regarded as operators at time* $x^0$

**Definition:** *The partial field of creation operators for a particle in interaction is* $|\underline{x\,,\alpha}\rangle$

$$\forall(x, \alpha) = \mathbb{R}^4 \otimes S, \quad |\underline{x\,,\alpha}\rangle: \mathbb{F} \to \mathbb{F} \tag{3.4.1}$$

It will be found that photons (and other vector particles) are not created in basis states $|x, \alpha\rangle$ of position. So in general $|\underline{x\,,\alpha}\rangle \neq |x\,,\alpha\rangle$ . and $|\underline{x\,,\alpha}\rangle$ must be found for each particle type.

**Definition:** *Let* $|\underline{\alpha}\rangle = |\underline{0,\alpha}\rangle$ *be the operator for the creation a particle at the origin.*

**Theorem:** The creation operator $|\underline{x\,,\alpha}\rangle: \mathbb{F} \to \mathbb{F}$ for a particle at $(x, \alpha) = \mathbb{R}^4 \otimes S$ is given by

$$|\underline{x\,,\alpha}\rangle = \sum_r \eta(r) \int_{\mathrm{M}} d^3\boldsymbol{p} \langle \boldsymbol{p}, r|\underline{\alpha}\rangle e^{ip \cdot x} |\boldsymbol{p}, r\rangle \tag{3.4.2}$$

*Proof:* By the resolution of unity, $|\underline{x\,,\alpha}\rangle: \mathbb{F} \to \mathbb{F}$ is given by

$$|\underline{x\,,\alpha}\rangle = \sum_r \eta(r) \int_{\mathrm{M}} d^3\boldsymbol{p} \langle \boldsymbol{p}, r|\underline{x\,,\alpha}\rangle |\boldsymbol{p}, r\rangle \tag{3.4.3}$$

The momentum space bound on the integral is needed because in RQG this is not a momentum space wave function with bounded support, but an operator on Hilbert space. Teleparallel space-time translation maps the creation operators appearing in interactions into each other. Then (3.4.2) follows by substituting

$$\langle \boldsymbol{p}, r|\underline{x\,,\alpha}\rangle = \langle \boldsymbol{p}, r|\underline{\alpha}\rangle e^{ip \cdot x} \tag{3.4.4}$$

**Definition:** *The derivative of the creation and annihilation operators is defined by formally differentiating (3.4.2).*

$$\partial|\underline{x,\alpha}\rangle = |\underline{\partial x,\alpha}\rangle = \sum_{r=0}^{3} \eta(r) \int_{\mathrm{M}} d^3\boldsymbol{p} \langle \boldsymbol{p}, r|\underline{\alpha}\rangle ip e^{ip \cdot x} |\boldsymbol{p}, r\rangle \tag{3.4.5}$$

There may be a number of different interactions, described by $I_j$: $\mathbb{F} \to \mathbb{F}$, where $j$ runs over an index set. Let $e_j \in \mathbb{R}$ be the coupling constant for the interaction $I_j$. Only one type of interaction takes place at a time, but there is uncertainty about which. Under the identification of addition with quantum logical OR, the interaction operator $I(x_0)$: $\mathscr{F} \to \mathscr{F}$, is

$$I = \sum_j e_j I_j$$

$I$ is Hermitian, and each $I_j$ is independent. So each $I_j$ is Hermitian (up to regularisation, (3.1.3)).

**Definition:** *In any finite discrete time interval,* T*, for each type of interaction, an operator* $H(x): \mathbb{F} \to \mathbb{F}$ *describes the interaction taking place at* $x = (x_0, \boldsymbol{x}) \in \mathrm{T} \otimes \mathrm{N}$*,* $H(x)$ *is called interaction density.*

The principle of homogeneity implies that $H(x)$ is the same, up to homomorphism, and has equal effect on a matter anywhere in N and for all times in T. $I_j$ describes equal certainty that a particle interacts anywhere in N. So by the identification of addition with quantum logical OR, $I_j$ can be written as a sum

$$I_j(x_0) = \sum_{\boldsymbol{x} \in \mathrm{N}} H(x_0, \boldsymbol{x}) = \sum_{\boldsymbol{x} \in \mathrm{N}} H(x) \tag{3.4.6}$$

The sum in (3.4.6) is over space, but not necessarily over spin. Without loss of generality $H(x)$ is Hermitian (up to normal ordering of the equal time product). By the definition of multiparticle space as a direct product, $H(x)$ can be factorised as a product of Hermitian operators, $J_\gamma(x)$, where $\gamma$ runs over the particles in the interaction

$$H(x) = \prod_{\gamma} J_\gamma(x) \tag{3.4.7}$$

**Definition:** *J is called a current operator* (its relationship to electric current will be shown).

A number of particles participate in the interaction. In the operator formalism participating particles prior to interaction are annihilated and particles present after interaction are created – a particle which is physically preserved is described as being annihilated and re-created. $H(x)$ can be represented as a Feynman node. Each line at the node corresponds to one particle in the interaction. In a single Feynman node there are no geometrical relationships with other matter, and it is not possible to say whether a particle's proper time is running forwards or backwards with respect to the reference frame clock. So a line for the annihilation of a particle, $\gamma$, may also represent the creation of the corresponding antiparticle $\bar{\gamma}$. The corresponding operator is called a field.

**Definition:** *Let* $\langle \underline{x , \alpha}|$ *be the annihilation operator for a particle, and let* $|\overline{x , \alpha}\rangle$ *be the creation operator for the antiparticle, at* $(x, \alpha) = (x_0, \boldsymbol{x}, \alpha) \in \mathrm{T} \otimes \mathrm{N}$. *Then the field* $\phi_\alpha(x) : \mathbb{F} \to \mathbb{F}$ *is*

$$\phi_\alpha(x) = |\overline{x , \alpha}\rangle + \langle \underline{x , \alpha}| \tag{3.4.8}$$

Clearly the Hermitian conjugate of a particle field is the antiparticle field

$$\phi^\dagger{}_\alpha(x) = |\underline{x , \alpha}\rangle + \langle \overline{x , \alpha}| \tag{3.4.9}$$

Each line at a Feynman node corresponds to a field modelling the creation or annihilation of a particle or antiparticle. In the general case $J_\gamma(x)$ is Hermitian so it combines the particle and antiparticle fields

$$J_\gamma(x) = J_\gamma(\phi_\alpha(x), \phi^\dagger{}_\alpha(x)) \tag{3.4.10}$$

Then the general form of the interaction is

$$I_j(x_0) = : \sum_{\boldsymbol{x} \in \mathrm{N}} \prod_{\gamma} J_\gamma(|\overline{x , \alpha}\rangle + \langle \underline{x , \alpha}|, |\underline{x , \alpha}\rangle + \langle \overline{x , \alpha}|) : \tag{3.4.11}$$

The colons denote normal ordering. Particular interactions can be postulated using operators with the general form of (3.4.11) and subject to the constraint that the resulting theoretical properties correspond to the observed behaviour of matter.

**Definition:** *Let* $\pi$ *be the permutation such that* $\tau_{\pi(n)} > \ldots \tau_{\pi(2)} > \tau_{\pi(1)}$ *Then the time ordered product is*

$$\mathrm{T}\{I(\tau_n)\ldots I(\tau_1)\} = I(\tau_{\pi(n)})\ldots I(\tau_{\pi(1)})$$

**Theorem:** (Locality)

$$\forall x, y \in \mathrm{T} \otimes \mathrm{N} \text{ such that } x - y \text{ is space-like } \langle [H(y), H(x)] \rangle = 0 \tag{3.4.12}$$

*Proof:* Iterate (3.1.2) from an initial condition at $t = 0$ given by $|f\rangle_0 \in \mathrm{F}$

$$|f\rangle_1 = \mu(1 - iI(0))|f\rangle_0$$

$$|f\rangle_2 = \mu^2(1 - iI(1))(1 - iI(0))|f\rangle_0$$

$$|f\rangle_3 = \mu^3(1 - iI(2))(1 - iI(1))(1 - iI(0))|f\rangle_0$$

Expand after $T$ iterations

$$|f\rangle_T = \mu^T \left( 1 + i \sum_{\tau_1 = 0}^{T-1} I(\tau) + (-i)^2 \sum_{\substack{\tau_2 = 0 \\ \tau_2 > \tau_1}}^{T-1} I(\tau_2) \sum_{\tau_1 = 0}^{T-1} I(\tau_1) + \ldots \right) |f\rangle_0 \qquad (3.4.13)$$

Then (3.4.13) is

$$|f\rangle_T = \mu^T \left( 1 + \sum_{n=1}^{T} \frac{(-i)^n}{n!} \sum_{\substack{\tau_1 \ldots \tau_n = 0 \\ i \neq j \Rightarrow \tau_i \neq \tau_j}}^{T-1} \mathrm{T}\{ I(\tau_n) \ldots I(\tau_1) \} \right) |f\rangle_0 \qquad (3.4.14)$$

There may be any number of particles in the initial state $|f\rangle_0 \in \mathbb{F}$ so (3.4.14) can be interpreted directly as a quantum logical statement meaning that, since an unknown number of interactions take place at undefined positions and undefined times, the final state is a weighted sum of hypotheticals.

In the information space interpretation the form of the field operator describes the possibility that creation/annihilation might be anywhere, not a quantised “matter field” which is, in some sense, everywhere. In this formulation the functional integral, or “sum over all paths” has as natural interpretation, not that a particle passes through all paths in space-time (as described by Feynman e.g. in [5]), but that the sum over paths is a weighted logical OR between the possible paths that might be detected if an experiment were done to trace the path.

Except for asymptotically free initial and final states, (3.4.14) ceases to make sense in the limit $T \to \infty$ and the expansion may reasonably be expected not to converge under such conditions, but there is no problem for bounded N and finite $T$ (i.e. stable fore and after states). By (3.4.6), (3.4.14) is

$$|f\rangle_T = \mu^T \left( 1 + \sum_{n=1}^{T} \frac{(-i)^n}{n!} \sum_{\substack{x^1 \ldots x^n \in \mathrm{T} \otimes \mathrm{N}_S \\ i \neq j \Rightarrow x_0^i \neq x_0^j}} \mathrm{T}\{ H(x^n) \ldots H(x^1) \} \right) |f\rangle_0 \qquad (3.4.15)$$

Under Lorentz transformation of (3.4.15) the order of interactions, $H(x^i)$, can be changed in the time ordered product whenever $x^i - x^j$ is space-like. This cannot affect the final state $|f\rangle_T$ for any $T \in \mathrm{N}$. So by (3.4.7) $H$ factorises and the locality condition applies to the current operators.

$$\forall x, y \in \mathrm{T} \otimes \mathrm{N}_S \textit{ such that } x - y \textit{ is space-like } \langle [J(y), J(x)] \rangle = 0 \qquad (3.4.16)$$

### 3.5 Conservation of Momentum

**Theorem:** In an inertial reference frame momentum is conserved.

*Remark:* This is related to Noether’s theorem and like it depends on invariance under space translation, but since we have not formulated the model from an action principle a separate demonstration is needed. Energy has been defined to be on mass shell and is not conserved in an individual interaction.

*Proof:* Classical momentum is the expectation of the momentum of a large number of particles. In the absence of interaction the expectation of momentum is constant for each particle by Newton's first law, (2.6.4). So it is sufficient to prove conservation of momentum in each particle interaction. Expand the interaction density, (3.4.11), as a sum of terms of the form

$$i(x_0) = \sum_{\boldsymbol{x} \in \mathrm{N}} h(x) = \sum_{\boldsymbol{x} \in \mathrm{N}} |\underline{x, \alpha}\rangle_1 \ldots |\underline{x, \alpha}\rangle_m \langle\underline{x, \alpha}|_{m+1} \ldots \langle\underline{x, \alpha}|_n \tag{3.5.1}$$

where $|\underline{x, \alpha}\rangle_i$ and $\langle\underline{x, \alpha}|_i$ are creation and annihilation operators for the particles and antiparticles in the interaction, given by (3.4.2). Suppress spin indices by writing $\forall \boldsymbol{p} \in \mathrm{M}$ $s = 1, 2, 3, 4$ $|\boldsymbol{p}\rangle = |\boldsymbol{p}, s\rangle$ and $|\underline{x}\rangle = |\underline{x, \alpha}\rangle$. From (3.5.1), $\forall n, m \in \mathrm{N}, n, m > 0$, $\forall$ plane wave $|\boldsymbol{p}^1\rangle, \ldots, |\boldsymbol{p}^n\rangle$

$$\begin{aligned}
&\langle \boldsymbol{p}^1; \ldots; \boldsymbol{p}^m | i(x_0) | \boldsymbol{p}^{m+1}; \ldots; \boldsymbol{p}^n \rangle \\
&\quad = \langle \boldsymbol{p}^1; \ldots; \boldsymbol{p}^m | \sum_{\boldsymbol{x} \in \mathrm{N}} |\underline{x}\rangle^1 \ldots |\underline{x}\rangle^m \langle\underline{x}|^{m+1} \ldots \langle\underline{x}|^n | \boldsymbol{p}^{m+1}; \ldots; \boldsymbol{p}^n \rangle \\
&\quad = \sum_{\boldsymbol{x} \in \mathrm{N}} \sum_{\pi} \varepsilon(\pi) \prod_{i=1}^{m} \langle \boldsymbol{p}^i | \underline{x} \rangle^{\pi(i)} \sum_{\pi'} \varepsilon(\pi') \prod_{j=m+1}^{n} \langle \underline{x} |^j \boldsymbol{p}^{\pi'(j)} \rangle
\end{aligned}$$

which is a sum of terms of the form

$$\sum_{\boldsymbol{x} \in \mathrm{N}} \prod_{i=1}^{m} \langle \boldsymbol{q}_i | \underline{x} \rangle_{\pi(i)} \prod_{j=m+1}^{n} \langle \underline{x} |_j \boldsymbol{p}_{\pi'(j)} \rangle \, .$$

Using (3.4.4) and permuting $\boldsymbol{p}_{\pi'(j)} \to \boldsymbol{p}_j$ this reduces to a sum of terms of the form

$$\begin{aligned}
&\sum_{\boldsymbol{x} \in \mathrm{N}} \prod_{i=1}^{m} \langle \boldsymbol{q}^i | \underline{\alpha} \rangle e^{iq^i \cdot x} \prod_{j=1}^{n} \langle \underline{\alpha} | \boldsymbol{p}^j \rangle e^{-ip^j \cdot x} \\
&\quad = \delta\left( \sum_{j=m+1}^{n} \boldsymbol{p}^j - \sum_{i=1}^{m} \boldsymbol{q}^i \right) \prod_{i=1}^{m} \langle \boldsymbol{q}^i | \underline{\alpha} \rangle e^{-iq_0^i x_0} \prod_{j=1}^{n} \langle \underline{\alpha} | \boldsymbol{p}_j \rangle e^{-p_0^j x_0}
\end{aligned}$$

Thus momentum is conserved in each particle interaction, and is conserved universally by Newton's first law (2.6.4).

### 3.6 Classical Law

We are interested in changes in classical observable quantities, or changes in the expectation, $\langle O \rangle$ of an observable, $O = O(x)$. Since measurement is a combination of interactions, observable quantities are composed of interaction operators, which, by (3.4.11), can be decomposed into fields. Then physically observable discrete values are obtained from differentiable functions, and difference equations in the discrete quantities are obtained by integrating differential equations over one unit of time.

**Lemma:** The equal time commutator between an observable $O$ such that $O(x) = O(H(x))$ and the interaction density, $H$, obeys the commutation relation

$$\forall \boldsymbol{x} \neq \boldsymbol{y}, [H(x), O(y)]_{x_0 = y_0} = 0 \tag{3.6.1}$$

*Proof:* Immediate from (3.4.16)

(3.6.1) involves the commutation relation between the interaction density, $H$, and the observable, $O$. Since all physical processes are described by interactions, any observable is a combination of interaction operators. Then observables are a combination of particle currents and (3.6.1) depends on the commutators for particle fields. For fermions the creation operators anticommute, but commutation relations are obtained if the current, (3.4.10), is a composition of an even number of fermion fields.

**Theorem:** In the flat space approximation the expectation of an observable $O(x) = O(H(x))$ obeys the differential equations

$$\begin{aligned} &\partial_0\langle O(x)\rangle = i\langle[H(x), O(x)]\rangle + \langle\partial_0 O(x)\rangle \\ &\text{For } \alpha = 1, 2, 3 \qquad \partial_\alpha\langle O(x)\rangle = \langle\partial_\alpha O(x)\rangle \end{aligned} \tag{3.6.2}$$

*Proof:* Since $|\mu|^2 = 1$ , $I = I^\dagger$ and $I^2(t) = 0$

$$\begin{aligned} \langle O(t+1)\rangle - \langle O(t)\rangle &= \langle f|_t(1+iI(t+1))O(t+1)(1-iI(t+1))|f\rangle_t - \langle O(t)\rangle \\ &= \langle f|_t i[I(t+1), O(t+1)] + O(t+1)|f\rangle_t - \langle f|_t O(t)|f\rangle_t \end{aligned}$$

Using linearity of kets treated as operators and rearranging

$$\langle O(t+1)\rangle - \langle O(t)\rangle = i\langle[I(t+1), O(t+1)]\rangle + \langle O(t+1) - O(t)\rangle \tag{3.6.3}$$

The solution to (3.6.3) is the restriction to integer values of the solution of

$$\partial_0\langle O(x)\rangle = i\langle[I(t), O(x)]\rangle + \langle\partial_0 O(x)\rangle \tag{3.6.4}$$

Using locality, (3.6.1), with $x_0 = y_0$ (3.6.4) is

$$\partial_0\langle O(x)\rangle = i\left\langle\left[\sum_{\boldsymbol{y}\in\mathrm{N}} H(x_0, \boldsymbol{y}), O(x)\right]\right\rangle + \langle\partial_0 O(x)\rangle \tag{3.6.5}$$

Using locality, (3.4.12), (3.6.5) reduces to the time-component of (3.6.2). The proof for $\alpha = 1, 2, 3$ is trivial.

**Corollary:** The classical observable $\langle O(x)\rangle$ is the expectation of an observable $O(x)$ in the limit of large sample behaviour. $\langle O(x)\rangle$ obeys the covariant differential equations

$$\begin{aligned} &\nabla_0\langle O(x)\rangle = i\langle[H(x), O(x)]\rangle + \langle\nabla_0 O(x)\rangle \\ &\text{For } \alpha = 1, 2, 3 \qquad \nabla_\alpha\langle O(x)\rangle = \langle\nabla_\alpha O(x)\rangle \end{aligned} \tag{3.6.6}$$

*Proof:* Repeat the demonstration of (3.6.2) taking into account that the time evolution of a classical observable is statistically determined, and hence the observable is effectively measured at all times. Then the motion may be treated as a sequence of small motions from initial to final state. For small displacements teleparallelism is the same as parallel transport and (3.6.6) reduces to (3.6.2).

Position is only a numerical value derived from a configuration of matter in measurement. This does not in itself require that particles are themselves sizeless, or point-like. But by (3.6.1) and (3.6.2) changes in $\langle O(x)\rangle$ have no dependence on interactions except at the point $x$, and this give physical meaning to the statement that elementary particles are point-like. By (3.6.2) $O(x)$ has no space-like dependence on particle interactions for any space-like slice. It follows that no observable particle effect may propagate faster than the speed of light.

# 4 Particle Fields

## 4.1 The Photon Field

Having zero mass the photon is its own antiparticle so that $|\overline{x, \alpha}\rangle = |\underline{x, \alpha}\rangle$ .

**Definition:** *By (3.4.8), the photon field is*

$$A_\alpha(x) = |\underline{x, \alpha}\rangle + \langle\underline{x, \alpha}| \tag{4.1.1}$$

This is Hermitian. So only one photon field is necessary in the current. Then $J = A$ is permissible and photons can be absorbed and emitted singly. Photons are bosons and the commutator is

$$[A_\alpha(x), A_\beta(y)] = [|\underline{x, \alpha}\rangle + \langle\underline{x, \alpha}|, |\underline{y, \beta}\rangle + \langle\underline{y, \beta}|] = \langle\underline{x, \alpha}|\underline{y, \beta}\rangle - \langle\underline{y, \beta}|\underline{x, \alpha}\rangle . \tag{4.1.2}$$

Thus, by (2.6.7) and (3.4.4),

$$[A_\alpha(x),A_\beta(y)] = \sum_r \eta(r)\int_M d^3\boldsymbol{p}(\langle\underline{\alpha}|\boldsymbol{p},r\rangle e^{-ip\cdot(x-y)} - \langle\underline{\beta}|\boldsymbol{p},r\rangle e^{ip\cdot(x-y)}\langle\boldsymbol{p},r|\underline{\alpha}\rangle)\,. \quad (4.1.3)$$

The constraint that $A_\alpha(x)$ has only components of spin $\alpha$ is necessary if the interaction operator creates eigenstates of spin. This is observed and we assume that it also holds for time-like and longitudinal spin. Then $\langle\underline{\alpha}|\boldsymbol{p},r\rangle$ transforms as $w_\alpha(\boldsymbol{p},r)$ (defined in (2.5.1)) under space inversion. So

$$\langle\underline{\beta}|-\boldsymbol{p},r\rangle\langle-\boldsymbol{p},r|\underline{\alpha}\rangle = \langle\underline{\alpha}|\boldsymbol{p},r\rangle\langle\boldsymbol{p},r|\underline{\beta}\rangle\,, \quad (4.1.4)$$

since $w_\alpha(\boldsymbol{p},0)$ has no space-like component and for $r = 1,2,3$ $w_\alpha(\boldsymbol{p},r)$ has no time like component. Now substitute $\boldsymbol{p}\to -\boldsymbol{p}$ in the second term of (4.1.3) at $x_0 = y_0$,

$$[A(x),A(y)]_{x_0=y_0} = 0\,. \quad (4.1.5)$$

Then by substituting $O = A$ in (3.6.2), and noting from (3.4.7) that the commutation relationship with the interaction density is determined by the commutation relationship with the current, find

$$\partial_\alpha\langle A_\beta(x)\rangle = \langle\partial_\alpha A_\beta(x)\rangle\,. \quad (4.1.6)$$

The physical interpretation of (4.1.6) is that observable effects associated with photons depend only on changes in photon number; since photons can be absorbed or emitted singly the number of photons cannot be an eigenstate of an operator constructed from the interaction and cannot be known. Let $\phi_\mu(x)$ be a gauge term, that is an arbitrary solution of

$$\partial_\mu\phi_\mu(x) = 0\,. \quad (4.1.7)$$

having no physical meaning. Then physical predictions from (4.1.6) are invariant under the gauge transformation $A(x)\to A(x)+\phi(x)$, and the value of $\langle A(x)\rangle$ is hidden by the gauge term. Differentiating (4.1.6) using (3.6.2) gives

$$\partial^2\langle A(x)\rangle = \partial_\alpha\langle\partial_\alpha A(x)\rangle = i\langle[H(x),\partial_0 A(x)]\rangle + \langle\partial^2 A(x)\rangle\,. \quad (4.1.8)$$

Observe that $p^2 = 0$ for the photon so $\partial^2|\underline{x,\alpha}\rangle = 0$. Then from (4.1.1)

$$\partial^2 A(x) = 0 \quad (4.1.9)$$

Then (4.1.8) reduces to

$$\partial^2\langle A(x)\rangle = i\langle[H(x),\partial_0 A(x)]\rangle\,. \quad (4.1.10)$$

Given $H$, (4.1.10) can be calculated from the commutator between the fields

$$[\partial A_\alpha(x),A_\beta(y)] = \langle\underline{\partial x,\alpha}|\underline{y,\beta}\rangle - \langle\underline{y,\beta}|\underline{\partial x,\alpha}\rangle\,. \quad (4.1.11)$$

But, by (3.4.5) and (3.4.4),

$$\langle\underline{\partial x,\alpha}|\underline{y,\beta}\rangle = -\sum_{r=0}^{3}\eta(r)\int_M d^3\boldsymbol{p}\langle\underline{\alpha}|\boldsymbol{p},r\rangle\langle\boldsymbol{p},r|\underline{\beta}\rangle ipe^{-ip\cdot(x-y)}\,, \quad (4.1.12)$$

and

$$\langle\underline{y,\beta}|\underline{\partial x,\alpha}\rangle = \sum_{r=0}^{3}\eta(r)\int_M d^3\boldsymbol{p}\langle\underline{\beta}|\boldsymbol{p},r\rangle\langle\boldsymbol{p},r|\underline{\alpha}\rangle ipe^{ip\cdot(x-y)}\,, \quad (4.1.13)$$

Substituting $\boldsymbol{p}\to-\boldsymbol{p}$ in (4.1.13) at $x_0 = y_0$ and using (4.1.4) and (4.1.11) gives,

for $i = 1,2,3$, $[\partial_i A(x),A(y)]_{x_0=y_0} = 0$, and, for the time-like component,

$$[\partial_0 A_\alpha(x), A_\beta(y)]_{x_0 = y_0} = -2i\sum_{r=0}^{3}\eta(r)\int_M d^3\boldsymbol{p}\langle\underline{\alpha}|\boldsymbol{p}, r\rangle\langle\boldsymbol{p}, r|\underline{\beta}\rangle p_0 e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \ . \tag{4.1.14}$$

**Theorem:** The equal time commutator, (4.1.14), satisfies locality, (3.4.16), if

$$\langle\underline{\alpha}|\boldsymbol{p}, r\rangle = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\frac{w_\alpha(\boldsymbol{p}, r)}{\sqrt{2p_0}} . \tag{4.1.15}$$

*Proof:* It follows from (4.1.15) that

$$\sum_{r=0}^{3}\eta(r)\langle\underline{\alpha}|\boldsymbol{p}, r\rangle\langle\boldsymbol{p}, r|\underline{\beta}\rangle = \frac{\eta(r)\delta_{\alpha\beta}}{16\pi^3 p_0} . \tag{4.1.16}$$

Substituting (4.1.16) into (4.1.14) shows locality is satisfied:

$$[\partial_0 A(x), A(y)]_{x_0 = y_0} = -ig\delta_{\boldsymbol{x}\boldsymbol{y}} . \tag{4.1.17}$$

Substituting (4.1.15) into (4.1.1) using (2.6.3) gives the photon field,

$$A_\alpha(x) = \sum_{r=0}^{3}\eta(r)\int_M\frac{d^3\boldsymbol{p}}{\sqrt{2p_0}}(e^{ip\cdot x}|\boldsymbol{p}, r\rangle + e^{-ip\cdot x}\langle\boldsymbol{p}, r|)w_\alpha(\boldsymbol{p}, r) . \tag{4.1.18}$$

By (4.1.16), (3.4.2) and (2.6.8)

$$\langle\underline{x, \alpha}|\underline{y, \beta}\rangle = \frac{g_{\alpha\beta}}{8\pi^3}\int\frac{d^3\boldsymbol{p}}{2p_0}e^{-ip\cdot(x-y)} . \tag{4.1.19}$$

So the commutator, (4.1.2), is

$$[A_\alpha(x), A_\beta(y)] = \frac{g_{\alpha\beta}}{8\pi^3}\int_M\frac{d^3\boldsymbol{p}}{2p_0}(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}) . \tag{4.1.20}$$

**Theorem:** (4.1.20) is zero outside the light cone.
*Proof:* The proof follows the text books, e.g. [1], and is left to the reader.
**Theorem:** $\langle A(x)\rangle$ satisfies the Lorentz gauge condition

$$\partial_\alpha\langle A_\alpha(x)\rangle = 0 . \tag{4.1.21}$$

*Proof:* By (4.1.6),

$$\begin{aligned}\partial_\alpha\langle A_\alpha(x)\rangle &= \langle\partial_\alpha A_\alpha(x)\rangle \\ &= \langle\sum_{r=0}^{3}\eta(r)\int_M\frac{d^3\boldsymbol{p}}{\sqrt{2p_0}}(e^{ip\cdot x}|\boldsymbol{p}, r\rangle + e^{-ip\cdot x}\langle\boldsymbol{p}, r|)i(p_\alpha - p_\alpha)w_\alpha(\boldsymbol{p}, r)\rangle ,\end{aligned}$$

by differentiating (4.1.18). But this is zero which establishes (4.1.21).

## 4.2 The Dirac Field

**Definition:** *By (3.4.8), the Dirac field is*

$$\psi_\alpha(x) = |\overline{x, \alpha}\rangle + \langle\underline{x, \alpha}| . \tag{4.2.1}$$

We know from observation that a Dirac particle can be an eigenstate of position. Any physical configuration can only be a combination of particle interactions so it is possible to form the position operator

$$Z(X) = \sum_{\boldsymbol{x}\in X}|\boldsymbol{x}\rangle\langle\boldsymbol{x}| \ . \tag{4.2.2}$$

from the current (3.4.10), for any region $X$ which can be as small as the apparatus will allow. Position kets are a basis. So (4.2.2) reduces to

$$Z(\boldsymbol{x}) = |\boldsymbol{x}\rangle\langle\boldsymbol{x}|$$

up to the resolution of the apparatus. Current can only generate eigenstates of spin and position if it does not mix basis states. So

$$\forall x \in N \;\; |\underline{x\,,\alpha}\rangle = |x\,,\alpha\rangle\,. \tag{4.2.3}$$

Then, by (2.6.1),

$$\langle\underline{\alpha}|\boldsymbol{p}\,,r\rangle = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} u_\alpha(\boldsymbol{p}, r)\,, \tag{4.2.4}$$

and by 18.8,

$$\langle\underline{x\,,\alpha}| = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \sum_r \int_M d^3\boldsymbol{p}\, u_\alpha(\boldsymbol{p}, r) e^{-ip\cdot x}\langle\boldsymbol{p}\,,r|\,. \tag{4.2.5}$$

**Definition:** *The Dirac adjoint of the annihilation operator* $\langle\underline{x\,,\alpha}|$ *is*

$$|\underline{x\,,\hat{\alpha}}\rangle = \sum_\mu |\underline{x,\mu}\rangle\gamma^0_{\mu\alpha} = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \sum_r \int_M d^3\boldsymbol{p}\,\hat{u}_\alpha(\boldsymbol{p}, r) e^{ip\cdot x}|\boldsymbol{p}\,,r\rangle\,. \tag{4.2.6}$$

Similarly by (2.6.2)

$$\langle\overline{\alpha}|\boldsymbol{p}\,,r\rangle = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \bar{v}_\alpha(\boldsymbol{p}, r)\,, \tag{4.2.7}$$

and by 18.8,

$$|\overline{x\,,\alpha}\rangle = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \sum_r \int_M d^3\boldsymbol{p}\, v_\alpha(\boldsymbol{p}, r) e^{ip\cdot x}|\boldsymbol{p}\,,r\rangle\,. \tag{4.2.8}$$

**Definition:** *The Dirac adjoint of the creation operator* $|\overline{x\,,\alpha}\rangle$ *is*

$$\langle\overline{x\,,\hat{\alpha}}| = \sum_\mu \langle\overline{x,\mu}|\gamma^0_{\mu\alpha} = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \sum_r \int_M d^3\boldsymbol{p}\,\hat{v}_\alpha(\boldsymbol{p}, r) e^{ip\cdot x}|\boldsymbol{p}\,,r\rangle\,. \tag{4.2.9}$$

**Definition:** *The Dirac adjoint of the field is*

$$\hat{\psi}_\alpha(x) = \psi^\dagger{}_\mu(x)\gamma^0_{\mu\alpha} = |\underline{x\,,\hat{\alpha}}\rangle + \langle\overline{x\,,\hat{\alpha}}|\,. \tag{4.2.10}$$

**Theorem:** The anticommutation relations for the Dirac field and Dirac adjoint and obey:

$$\{\psi_\nu(x), \psi_\lambda(y)\} = \{\hat{\psi}_\mu(x), \hat{\psi}_\kappa(y)\} = 0\,, \tag{4.2.11}$$

$$\{\psi_\alpha(x), \hat{\psi}_\beta(y)\}_{x_0 = y_0} = \gamma^0_{\alpha\beta}\delta_{\boldsymbol{xy}}\,. \tag{4.2.12}$$

*Proof:* (4.2.11) follows from the definitions, (4.2.1) and (4.2.10). We have

$$\begin{aligned}\{\psi_\alpha(x), \hat{\psi}_\beta(y)\} &= \{\langle\underline{x\,,\alpha}|, |\underline{y,\hat{\beta}}\rangle\} + \{|\overline{x\,,\alpha}\rangle, \langle\overline{y,\hat{\beta}}|\}\\ &= \langle\underline{x\,,\alpha}|\underline{y,\hat{\beta}}\rangle + \langle\overline{y,\hat{\beta}}|\overline{x\,,\alpha}\rangle^{\mathrm{T}}\,,\end{aligned} \tag{4.2.13}$$

where $^{\mathrm{T}}$ denotes that $\alpha$ and $\beta$ are transposed. By (4.2.5) and (4.2.6), and using (2.6.8),

$$\begin{aligned}\langle\underline{x\,,\alpha}|\underline{y,\hat{\beta}}\rangle &= \frac{1}{8\pi^3}\sum_r \int_M d^3\boldsymbol{p}\, u_\alpha(\boldsymbol{p}, r)\hat{u}_\beta(\boldsymbol{p}, r) e^{-ip\cdot(x-y)}\\ &= \frac{1}{8\pi^3}\int_M \frac{d^3\boldsymbol{p}}{2p_0}(p\cdot\gamma + m)_{\alpha\beta} e^{-ip\cdot(x-y)}\,.\end{aligned} \tag{4.2.14}$$

Likewise for the antiparticle, by (4.2.8) and (4.2.9),

$$\langle \overline{y, \hat{\beta}} | \overline{x, \alpha} \rangle^{\mathrm{T}} = \frac{1}{8\pi^3} \sum_r \int_M d^3\boldsymbol{p}\; v_\alpha(\boldsymbol{p}, r) \hat{v}_\beta(\boldsymbol{p}, r)\; e^{ip \cdot y - ix \cdot p}$$

$$= \frac{1}{8\pi^3} \int_M \frac{d^3\boldsymbol{p}}{2p_0} (p \cdot \gamma - m)_{\alpha\beta}\; e^{ip \cdot (x-y)} . \tag{4.2.15}$$

Substituting $\boldsymbol{p} \to -\boldsymbol{p}$ at $x_0 = y_0$ in (4.2.15) gives

$$\langle \overline{y, \hat{\beta}} | \overline{x, \alpha} \rangle_{x_0 = y_0} = \frac{1}{8\pi^3} \int_M \frac{d^3\boldsymbol{p}}{2p_0} (2p_0 \gamma^0 - p \cdot \gamma - m) e^{-i\boldsymbol{p} \cdot (\boldsymbol{x} - \boldsymbol{y})} . \tag{4.2.16}$$

So, by (4.2.13), adding (4.2.14) and (4.2.16) at $x_0 = y_0$ gives the equal time anticommutator,

$$\{\psi_\alpha(x), \hat{\psi}_\beta(y)\}_{x_0 = y_0} = \frac{1}{8\pi^3} \gamma^0_{\alpha\beta} \int_M d^3\boldsymbol{p}\; e^{-i\boldsymbol{p} \cdot (\boldsymbol{x} - \boldsymbol{y})} . \tag{4.2.17}$$

(4.2.12) follows immediately.

**Theorem:** The anticommutation relation for the Dirac field and the Dirac adjoint is zero outside the light cone.

*Proof:* By (4.2.14)

$$\langle \underline{x, \alpha} | \underline{y, \hat{\beta}} \rangle = \frac{1}{8\pi^3} (i\partial \cdot \gamma + m) \int_M \frac{d^3\boldsymbol{p}}{2p_0} e^{-ip \cdot (x-y)} . \tag{4.2.18}$$

And by (4.2.15)

$$\langle \overline{y, \hat{\beta}} | \overline{x, \alpha} \rangle^{\mathrm{T}} = -\frac{1}{8\pi^3} (i\partial \cdot \gamma + m) \int_M \frac{d^3\boldsymbol{p}}{2p_0}\; e^{ip \cdot (x-y)} . \tag{4.2.19}$$

By (4.2.13) the anticommutator is found by adding (4.2.18) and (4.2.19)

$$\{\psi_\alpha(x), \hat{\psi}_\beta(y)\} = \frac{1}{8\pi^3} (i\partial \cdot \gamma + m) \int_M \frac{d^3\boldsymbol{p}}{2p_0} (e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}) . \tag{4.2.20}$$

(4.2.20) is and zero outside the light cone. The proof follows the text books, e.g. [1], and is left to the reader.

## 5 Electrodynamics

### 5.1 The Electric Current

QED is defined by the intuitively appealing minimal interaction characterised by the emission or absorption of a photon by a Dirac particle. According to (3.4.7) an interaction $H$ between photons and Dirac particles is described by a combination of particle currents, which, by (3.4.10), are themselves Hermitian combinations of particle fields.

**Definition:** *The Dirac current operator is*

$$j_\alpha(x) = :\hat{\psi}_\mu(x) \gamma^\alpha_{\mu\nu} \psi_\nu(x): = :\hat{\psi}(x) \gamma^\alpha \psi(x): . \tag{5.1.1}$$

**Definition:** *Let e be the electromagnetic coupling constant. The electromagnetic interaction density is*

$$H(x) = ej(x) \cdot A(x) = e:\hat{\psi}(x) \gamma \cdot A(x) \psi(x): . \tag{5.1.2}$$

**Lemma:**

$$\langle \partial \cdot j(x) \rangle = 0 . \tag{5.1.3}$$

*Proof:* Using the definitions (4.2.1) and (4.2.10) to expand (5.1.1),

$$j_\alpha(x) = |\underline{x,\hat{\mu}}\rangle\gamma^\alpha_{\mu\nu}|\overline{x,\nu}\rangle + |\underline{x,\hat{\mu}}\rangle\gamma^\alpha_{\mu\nu}\langle\underline{x,\nu}| - \gamma^\alpha_{\mu\nu}|\overline{x,\nu}\rangle\langle\overline{x,\hat{\mu}}| + \langle\overline{x,\hat{\mu}}|\gamma^\alpha_{\mu\nu}\langle\underline{x,\nu}| \,, \tag{5.1.4}$$

where the summation convention is used for the repeated indices, μ and ν. In classical situations we only consider states of a definite number of Dirac particles. So the expectation of the pair creation and annihilation terms is zero. Using (4.2.5) and (4.2.6) and differentiating the particle term in (5.1.4),

$$\partial_\alpha|\underline{x,\hat{\mu}}\rangle\gamma^\alpha_{\mu\nu}\langle\underline{x,\nu}| \,.$$

$$= \frac{1}{8\pi^3}\sum_{r,s}\int_M d^3\boldsymbol{p}\int_M d^3\boldsymbol{q}\, i\hat{u}(\boldsymbol{p},r)(q\cdot\gamma - p\cdot\gamma)u(\boldsymbol{q},s)e^{ix\cdot(q-p)}|\boldsymbol{p},r\rangle\langle\boldsymbol{q},s|$$

Using (4.2.8) and (4.2.9) and differentiating the antiparticle term in (5.1.4),

$$\partial_\alpha\gamma^\alpha_{\mu\nu}|\overline{x,\nu}\rangle\langle\overline{x,\hat{\mu}}|$$

$$= \frac{1}{8\pi^3}\sum_{r,s}\int_M d^3\boldsymbol{p}\int_M d^3\boldsymbol{q}\, i\hat{v}(\boldsymbol{q},r)(p\cdot\gamma - q\cdot\gamma)v(\boldsymbol{p},s)e^{ix\cdot(p-q)}|\boldsymbol{p},r\rangle\langle\boldsymbol{q},s| \,.$$

Here $v$ and $\hat{v}$ have been ordered so that the spin index can be unambiguously omitted. (5.1.3) follows by differentiating (5.1.4) and using standard properties of Dirac spinors.

**Lemma:**

$$[j_0(x), j_\alpha(x)] = 0 \,. \tag{5.1.5}$$

*Proof:*

$$[\psi(x), j_\alpha(x)] = [\psi(x), :\hat{\psi}(x)\gamma^\alpha\psi(x):] = \{\psi(x), \hat{\psi}(x)\}\gamma^\alpha\psi(x) = \gamma^0\gamma^\alpha\psi(x) \tag{5.1.6}$$

by (4.2.12). Take the Hermitian conjugate:

$$[j_\alpha(x), \psi^\dagger(x)] = \psi^\dagger(x)\gamma^{\alpha\dagger}\gamma^0 = \hat{\psi}(x)\gamma^\alpha \,.$$

Postmultiply by $\gamma^0$:

$$[j_\alpha(x), \hat{\psi}(x)] = \hat{\psi}_\mu(x)\gamma^\alpha\gamma^0 \tag{5.1.7}$$

So by commuting the terms and using (5.1.6) and (5.1.7),

$$\begin{aligned} [j_0(x), j_\alpha(x)] &= [:\hat{\psi}(x)\gamma^0\psi(x):, j_\alpha(x)] \\ &= \hat{\psi}(x)\gamma^0[\psi(x), j_\alpha(x)] + [\hat{\psi}(x), j_\alpha(x)]\gamma^0\psi(x) \\ &= \hat{\psi}(x)\gamma^0\gamma^0\gamma^\alpha\psi(x) - \hat{\psi}(x)\gamma^\alpha\gamma^0\gamma^0\psi(x) = 0 \,. \end{aligned}$$

**Theorem:** $\langle j\rangle$ is a classical conserved current:

$$\partial \cdot \langle j(x)\rangle = 0 \,. \tag{5.1.8}$$

*Proof:* Substituting $O = j_\alpha$ in (3.6.2)

$$\partial_a\langle j_\alpha(x)\rangle = i\langle[H(x), j_0(x)]\rangle + \langle\partial_\alpha j_\alpha(x)\rangle \,.$$

(5.1.8) follows from (5.1.3) and (5.1.5).

**Theorem:** $\langle j_0\rangle$ can be identified with classical electric charge density:

$$\forall|f\rangle \in \mathbb{F},\ \langle j_0(x)\rangle = |\langle x|f\rangle|^2 - |\langle f|\overline{x}\rangle|^2 \,. \tag{5.1.9}$$

*Proof:* It is sufficient to show the theorem for a one particle state $|f\rangle \in \mathbb{H}$. By (5.1.4),

$$\begin{aligned} \langle j_0(x)\rangle &= \langle f|\underline{x,\hat{\mu}}\rangle\gamma^0_{\mu\nu}\langle\underline{x,\nu}|f\rangle - \gamma^0_{\mu\nu}\langle f|\overline{x,\nu}\rangle\langle\overline{x,\hat{\mu}}|f\rangle \\ &= \langle f|\underline{x}\rangle\gamma^0\gamma^0\langle\underline{x}|f\rangle - \langle\overline{x}|f\rangle\gamma^0\gamma^0\langle f|\overline{x}\rangle \,, \end{aligned}$$

where the terms are ordered so that the spinor indices can be suppressed. Then (5.1.9) follows at once.

### 5.2 Maxwell's Equations

Because classical law does not form part of the assumptions the claim that the minimal interaction is the cause of the electromagnetic force requires:

**Theorem:** $\langle A(x)\rangle$ satisfies Maxwell's Equations in Lorentz gauge:

$$\partial^2\langle A(x)\rangle = -e\langle j(x)\rangle . \tag{5.2.1}$$

*Proof:* Lorentz gauge was established in (4.1.21). Then, by (4.1.10) and (5.1.2),

$$\partial^2\langle A(x)\rangle = i\langle [j(x)\cdot A(x), \partial_0 A(x)]\rangle .$$

(5.2.1) follows immediately from (4.1.17).

### 5.3 Feynman Rules

**Definition:** *For any vector $p$ such that $p^2 = m^2$, and for $\tilde{p}_0 \in \mathbb{R}$ let $\tilde{p} = (\tilde{p}_0, \boldsymbol{p})$ be a matrix. $\tilde{p}$ satisfies the identity:*

$$\tilde{p}_0^2 - p_0^2 \equiv \tilde{p}^2 - m^2 . \tag{5.3.1}$$

**Lemma:** For $x > 0$, $\varepsilon > 0$ we have the identities:

$$\frac{e^{i(p_0 - i\varepsilon)x}}{2(p_0 - i\varepsilon)} \equiv \frac{-i}{2\pi}\int_{-\infty}^{\infty} d\tilde{p}_0 \frac{e^{-i\tilde{p}_0 x}}{\tilde{p}_0^2 - (p_0 - i\varepsilon)^2} \equiv \frac{-i}{2\pi}\int_{-\infty}^{\infty} d\tilde{p}_0 \frac{e^{-i\tilde{p}_0 x}}{\tilde{p}^2 - m^2 + 2ip_0\varepsilon + \varepsilon^2}, \tag{5.3.2}$$

$$\frac{e^{i(p_0 - i\varepsilon)x}}{2} \equiv \frac{-i}{2\pi}\int_{-\infty}^{\infty} d\tilde{p}_0 \frac{\tilde{p}_0 e^{-i\tilde{p}_0 x}}{\tilde{p}^2 - m^2 + 2ip_0\varepsilon + \varepsilon^2} . \tag{5.3.3}$$

and for $x < 0$ $\varepsilon > 0$ we have the identities

$$\frac{e^{-i(p_0 - i\varepsilon)x}}{2(p_0 - i\varepsilon)} \equiv \frac{-i}{2\pi}\int_{-\infty}^{\infty} d\tilde{p}_0 \frac{e^{-i\tilde{p}_0 x}}{\tilde{p}^2 - m^2 + 2ip_0\varepsilon + \varepsilon^2}, \tag{5.3.4}$$

$$\frac{e^{-i(p_0 - i\varepsilon)x}}{2} \equiv \frac{-i}{2\pi}\int_{-\infty}^{\infty} d\tilde{p}_0 \frac{\tilde{p}_0 e^{-i\tilde{p}_0 x}}{\tilde{p}^2 - m^2 + 2ip_0\varepsilon + \varepsilon^2} . \tag{5.3.5}$$

*Proof:* These are evaluated as contour integrals and the proofs are left to the reader.

**Definition:** *The step function is given $\forall x \in \mathbb{R}$, by*

$$\Theta(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Let $|g\rangle \in \mathbb{F}$ be a measured state at time $T$. $\langle g|f\rangle_T$ can be evaluated iteratively from (3.4.15). The result is the sum of the terms generated by the braket between $\langle \underline{x^n, \alpha}|$ and every earlier creation operator $|\underline{x^j, \alpha}\rangle$ and every particle in $|f\rangle_0$, and the braket between $|\underline{x^n, \alpha}\rangle$ and every later annihilation operator $\langle \underline{x^j, \alpha}|$ and every particle in the final state $\langle g|$ (all other brakets are zero). This procedure is repeated for every creation and annihilation operator in (5.1.4), and for every term in (3.4.15). To keep check on the brakets so formed, each factor $I_j(x_0)$ in (3.4.11) is represented as a Feynman node. Each line at the node corresponds to one of the particles in the interaction and to one of the particle fields $|\overline{x, \alpha}\rangle + \langle \underline{x, \alpha}|$ in $I_j(x_0)$. Then, when the braket is formed, the corresponding connection between the nodes is made in a diagram. Each internal connecting line, or propagator, is associated with a particular particle type. Photons are denoted by wavy lines, and Dirac particles by arrowed lines such that for particles the arrow is in the direction of time ordering in (3.4.15), and for antiparticles the arrow is opposed to the time ordering. In this way all time ordered diagrams are formed by making each possible connection, from the creation of a particle to the annihilation of a particle of the same type, and we calculate rules to evaluate the diagram from (3.4.15). There is an overall factor $1/n!$ for a diagram with $n$ vertices. The vertices, $x^n$, are such that $n \neq j \Rightarrow x_0^n \neq x_0^j$ and, by examination of (3.4.15) and (5.1.2), generate the expression

$$e \sum_{x^n \in \mathrm{T} \otimes \mathrm{N}} -i\gamma \,. \tag{5.3.6}$$

The initial and final states are expressed as plane wave expansions so that the time invariant inner product, (2.6.7), can be used so that μ in (3.4.15) is set to 1. Plane waves span $\mathbb{F}$, so can be used for the initial and final states without loss of generality. Then for an initial particle the state $|\boldsymbol{p}, r\rangle$ connected to the node $x^n$ gives, from (3.4.4),

$$\langle \underline{x^n, \alpha} | \boldsymbol{p}, r\rangle = \langle \underline{\alpha} | \boldsymbol{p}, r\rangle e^{-ip \cdot x^n} \,.$$

So for a photon, by (4.1.15),

$$\langle \underline{x^n, \alpha} | \boldsymbol{p}, r \rangle = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \frac{w_\alpha(\boldsymbol{p}, r)}{\sqrt{2p_0}} e^{-ip \cdot x^n} \,; \tag{5.3.7}$$

for a Dirac particle, by (4.2.4),

$$\langle \underline{x^n, \alpha} | \boldsymbol{p}, r \rangle = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} u_\alpha(\boldsymbol{p}, r) e^{-ip \cdot x^n} \,;$$

and for an antiparticle, by (4.2.7),

$$\langle \overline{x^n, \hat{\alpha}} | \boldsymbol{p}, r \rangle = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \hat{v}_\alpha(\boldsymbol{p}, r) e^{-ip \cdot x^n} \,. \tag{5.3.8}$$

Similarly for final particles in the state $\langle \boldsymbol{p}, r|$ connected to the node $x^n$ gives for photon, Dirac particle and antiparticle respectively

$$\left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \frac{w_\alpha(\boldsymbol{p}, r)}{\sqrt{2p_0}} e^{ip \cdot x^n} ,\ \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \hat{u}_\alpha(\boldsymbol{p}, r) e^{ip \cdot x^n} \text{ and } \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} v_\alpha(\boldsymbol{p}, r) e^{ip \cdot x^n} \,. \tag{5.3.9}$$

The time ordered product in (3.4.15) leads to an expression for the photon propagator

$$\Theta(x_0^n - x_0^j)\langle \underline{x^n, \alpha} | \underline{x^j, \beta}\rangle + \Theta(x_0^j - x_0^n)\langle \underline{x^j, \beta} | \underline{x^n, \alpha}\rangle^{\mathrm{T}} \,. \tag{5.3.10}$$

(5.3.10) is to be compared with the decomposition of distributions into advanced and retarded parts according to the method of Epstein and Glaser which also excludes $x_0^n = x_0^j$ [17][3]. Indeed our analysis of the origin of the ultraviolet divergence is essentially the same as that given by Scharf [17]. The difference between this treatment and Scharf is that our limiting procedure uses a discrete space and here the "switching off and switching on" of the interaction at $x_0^n = x_0^j$ is a physical constraint meaning that only one interaction takes place for each particle in any instant, as discussed above. By (4.1.19) (5.3.10) is

$$\frac{g_{\alpha\beta}}{8\pi^3} \int_{\mathrm{M}} \frac{d^3\boldsymbol{p}}{2p_0} [\Theta(x_0^n - x_0^j) e^{-ip \cdot (x^n - x^j)} + \Theta(x_0^j - x_0^n) e^{ip \cdot (x^n - x^j)}] \,.$$

Use (5.3.2) in the first term, recalling that $m^2 = 0$, and use (5.3.4) and substitute $\boldsymbol{p} \to -\boldsymbol{p}$ in the second term. Then we have

$$-i\frac{g_{\alpha\beta}}{16\pi^4} \int_{\mathrm{M}} \frac{d^3\boldsymbol{p}}{2p_0} \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} d\tilde{p}_0 [\Theta(x_0^n - x_0^j) + \Theta(x_0^j - x_0^n)] \frac{e^{-i\tilde{p} \cdot (x^n - x^j)}}{\tilde{p}^2 + 2ip_0\varepsilon + \varepsilon^2} \,. \tag{5.3.11}$$

For each node the Dirac current generates two propagators, one for the field and one for the adjoint. The field either annihilates a particle or creates an antiparticle, and is represented by an arrowed line pointing towards the vertex. The field $\psi_\alpha(x^n)$ at vertex $n$ acting on vertex $j$, generates the propagator arrowed from $j$ to $n$:

$$\Theta(x_0^n - x_0^j)\langle \underline{x^n, \alpha} | \underline{x^j, \hat{\beta}}\rangle - \Theta(x_0^j - x_0^n)\langle \overline{x^j, \hat{\beta}} | \overline{x^n, \alpha}\rangle^{\mathrm{T}} . \tag{5.3.12}$$

The Dirac adjoint field creates a particle or annihilates an antiparticle, and is represented by an arrowed line pointing away from the vertex. The adjoint $\hat{\psi}_\alpha(x^n)$ generates the propagator arrowed from $n$ to $j$:

$$\Theta(x_0^n - x_0^j)\langle \overline{x^n, \hat{\alpha}} | \overline{x^j, \beta}\rangle - \Theta(x_0^j - x_0^n)\langle \underline{x^j, \beta} | \underline{x^n, \hat{\alpha}}\rangle^{\mathrm{T}} . \tag{5.3.13}$$

The time ordered product in (3.4.15) is unaffected under the interchange of $(x^n, \alpha)$ and $(x^j, \beta)$. By interchanging $(x^n, \alpha)$ and $(x^j, \beta)$, we find, for the adjoint propagator arrowed from $j$ to $n$,

$$\Theta(x_0^j - x_0^n)\langle \overline{x^j, \hat{\beta}} | \overline{x^n, \alpha}\rangle^{\mathrm{T}} + \Theta(x_0^n - x_0^j)\langle \underline{x^n, \alpha} | \underline{x^j, \hat{\beta}}\rangle \tag{5.3.14}$$

(5.3.14) is identical to (5.3.12), the expression for the Dirac propagator arrowed from $j$ to $n$, and we do not distinguish whether an arrowed line in a diagram is generated by the field or the adjoint field. Similarly we find that the photon propagator, (5.3.10), is unchanged under interchange of the nodes. So we identify all diagrams which are the same apart from the ordering of the vertices and remove the overall factor $1/n!$ for a diagram with $n$ vertices. By (4.2.14) and (4.2.15), (5.3.12) is

$$\frac{\Theta(x_0^n - x_0^j)}{8\pi^3}\int_M \frac{d^3\boldsymbol{p}}{2p_0}(ip\cdot\gamma + m)e^{-ip\cdot(x^n - x^j)} + \frac{\Theta(x_0^j - x_0^n)}{8\pi^3}\int_M \frac{d^3\boldsymbol{p}}{2p_0}(ip\cdot\gamma - m)e^{ip\cdot(x^n - x^j)} . \tag{5.3.15}$$

Use (5.3.2) and (5.3.3) in the first term, and use (5.3.4) and (5.3.5) and substitute $\boldsymbol{p} \to -\boldsymbol{p}$ in the second term. Then the propagator (5.3.15) is

$$-i\frac{g_{\alpha\beta}}{16\pi^4}\int_M \frac{d^3\boldsymbol{p}}{2p_0}\lim_{\varepsilon \to 0^+}\int_{-\infty}^{\infty} d\tilde{p}_0[\Theta(x_0^n - x_0^j) + \Theta(x_0^j - x_0^n)]\frac{(ip\cdot\gamma + m)e^{-i\tilde{p}\cdot(x^n - x^j)}}{\tilde{p}^2 - m^2 + 2ip_0\varepsilon + \varepsilon^2} .$$

Now collect all the exponential terms with $\boldsymbol{x}^n$ in the exponent under the sum (5.3.6), and observe that the sum over space is a momentum conserving delta function. Then integrate over momentum space and impose conservation of momentum at each vertex, leaving for each independent internal loop

$$\frac{1}{8\pi^3}\int_M \frac{d^3\boldsymbol{p}}{2p_0} . \tag{5.3.16}$$

Only the time component remains in the exponents for the external lines (5.3.7) - (5.3.9). Introduce a finite cutoff $\Lambda \in \mathrm{N}$ by writing the improper integral

$$\frac{-i}{2\pi}\int_{-\infty}^{\infty} d\tilde{p}_0 = \lim_{\Lambda \to \infty}\int_{-\Lambda\pi}^{\Lambda\pi} d\tilde{p}_0 ,$$

and instructing that the limits $\Lambda \to \infty$, $\varepsilon \to 0^+$ should be taken after calculation of all formulae. Then the photon propagator, (5.3.11) reduces to

$$-\frac{ig_{\alpha\beta}}{2\pi}\int_{-\Lambda\pi}^{\Lambda\pi} d\tilde{p}_0\frac{(1 - \delta_{x_0^n x_0^j})e^{i\tilde{p}_0(x_0^n - x_0^j)}}{\tilde{p}^2 + 2ip_0\varepsilon + \varepsilon^2} \tag{5.3.17}$$

For a Dirac particle, $p_0 > 0$, and we can simplify the denominator by shifting the pole under the limit $\varepsilon \to 0^+$ and replacing $2ip_0\varepsilon + \varepsilon^2$ with $i\varepsilon$. Thus the Dirac propagator arrowed from $j$ to $n$ is

$$\frac{-i}{2\pi}\int_{-\Lambda\pi}^{\Lambda\pi} d\tilde{p}_0 \frac{(1-\delta_{x_0^n x_0^j})(\tilde{p}\cdot\gamma + m)_{\alpha\beta} e^{-i(x_0^n - x_0^j)\tilde{p}_0}}{\tilde{p}^2 - m^2 + i\varepsilon} \tag{5.3.18}$$

The propagators, (5.3.17) and (5.3.18), vanish for $x_0^j = x_0^n$, and are finite in the limit $\Lambda \to \infty$, since the integrands oscillate and tend to zero as $p_0 \to \infty$. Loop integrals, (5.3.16), are proper and the denominators do not vanish so the ultraviolet divergence and the infrared catastrophe are absent if the limits $\Lambda \to \infty$ and $\varepsilon \to 0^+$ are not taken prematurely. In the denominator of (5.3.17), $\varepsilon^2$ plays the role of the small photon mass commonly used to treat the infrared catastrophe, and, as with Scharf's treatment, there is no additional requirement to include a photon mass. The standard rules are obtained by ignoring $\delta_{x_0^n x_0^j}$ in the numerator of (5.3.17) and (5.3.18), and observing that for $\Lambda \in \mathrm{N}$, the sums over time in the vertex (5.3.6) act as $\tilde{p}_0$ conserving $\delta$ functions. Energy was defined in RQG to be the time component of a vector in a measured state, and so on mass shell, but it is now natural to extend the definition so that the conserved quantity, $\tilde{p}_0$, is also called energy in unobserved states, in which case the particle is said to be off mass shell.

Thus the discrete theory modifies the standard rules for the propagators and justifies the subtraction of divergent quantities, but here this is no ad hoc procedure but regularisation by the subtraction of a term which recognises that a particle cannot be annihilated at the instant of its creation. To see that this subtraction is effectively the same as that described by Scharf we replace $\Theta$ in the propagators (5.3.10) and (5.3.12) with a monotonous $C^\infty$ function $\chi_0$ over $\mathrm{R}^1$ with

$$\chi_0(t) = \begin{cases} 0 & \text{for } t \le 0 \\ 1 & \text{for } t \ge \chi \end{cases}$$

Then we observe that in the limit as the discrete unit of time $\chi \to 0$ the sums over space become integrals since these are well defined [17]. In the limit M is replaced by $\mathbb{R}^3$ and (5.3.10) and (5.3.12) are replaced by distributions which have been split with causal support. When the distributions are combined in (5.3.17) and (5.3.18) we obtain the usual Feynman rules, together with a term coming from $\delta_{x_0^n x_0^j}$, which gives a distribution with point support in the limit (c.f. Scharf [17] 3.2.46). The most straightforward way to determine the effect of this term is to consider the non-perturbative solution. This allows us to impose regularisation conditions on the propagator at low energies, that it is independent of lattice spacing $\chi$ to first order, and that the renormalised mass and charge adopt their bare values, since bare mass and charge appear in Maxwell's equations.

This in no way contradicts the calculation that the apparent or "running" coupling constant exhibits an energy dependency in scattering due to perturbative corrections. But it shows that this dependency is removed in the calculation of the expectation of the current, and enables us to regularise the theory to the low energy value. The calculation of effective charge [15].(7.96) by the summation of one particle irreducible insertions [15].(7.94) into the photon propagator breaks down to any finite order, so that the limit may not be taken. More generally the renormalisation group arguments leading to the behaviour of the Callan-Symanzik equation depend upon the sum of a geometric series [15].(10.27) which does not converge at the Landau pole. The Landau pole is absent in a model in which there is a fundamental minimum unit of time since high energies correspond to short interaction times and in a discrete model this implies fewer interactions.

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